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# Nested Witt vectors and their $q$-deformation ☆

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**Abstract**

Let $N \subseteq \mathbb{N}$ be a truncation set. We study the ring of $N$-nested Witt vectors and its $q$-deformation. Given two arbitrary integers $q$ and $r$, we provide a necessary and sufficient condition of $A$ so that $\mathbb{W}_N^q(A)$ and $\mathbb{W}_N^r(A)$ should be strictly isomorphic to each other. Also, an isomorphism of functors, $\mathbb{W}_N^q \circ \mathbb{W}_M^q \cong \mathbb{W}_{MN}^q$, will be established for coprime truncation sets $M$ and $N$. As a byproduct, we deal with interesting connections between nested Witt-vectors and other areas such as generalized Möbius $\mu$-functions and numerical polynomials, i.e., polynomials which take integral values at integer arguments.


*Keywords:* Witt vectors; Necklace ring; Möbius inversion function

## 1. Introduction

Let $\mathbb{N}$ be the set of positive integers, and let $\emptyset \neq N \subseteq \mathbb{N}$ be a *truncation set*, i.e., $N$ contains every positive divisor of each of its elements. Classically, one has for every commutative ring $A$ the associated ring of Witt vectors $\mathbb{W}(A)$. One can generalize this construction to elicit the notion of the ring of $N$-nested Witt vectors $\mathbb{W}_N(A)$ by restricting the index set from $\mathbb{N}$ to $N$.

The first example of nested Witt vectors may be the $p$-typical Witt vectors. More precisely, in 1936, E. Witt [16] considered the ring of nested Witt vectors corresponding to the set $\{1, p, p^2, \ldots\}$, where $p$ is any prime, with a view to describing complete discrete valuation rings analogous to the ring of $p$-adic integers. Around 1965, Witt together with S. Lang [6] introduced the concept of the ring of Witt vectors by considering the full truncation set, that is, the

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set of natural numbers. Another significant nested Witt vectors appeared under the name of the Witt–Burnside ring of a finite cyclic group in Dress and Siebeneicher' paper [3]. In this case, the corresponding truncation set coincides with the set of divisors of the order of the group under consideration.

An interesting phenomenon related with the ring of nested Witt vectors is that it has a $q$-deformation. More precisely, for every integer $q$ and every commutative ring $A$, one has the ring of $q$-deformed Witt vectors $\mathbb{W}_N^q(A)$. When $q=1$, it coincides with $\mathbb{W}_N(A)$. Furthermore, when $q=0$, the corresponding ring $\mathbb{W}_N^0(A)$ turns out to have quite a simple structure (see Section 2). As far as this $q$-deformation is concerned, we study two fundamental problems. Let $q$ and $r$ be arbitrary integers. The first problem is to find a necessary and sufficient condition on $A$ so that $\mathbb{W}_N^q(A)$ and $\mathbb{W}_N^r(A)$ be strictly isomorphic to each other. More precisely, we will provide the following criterion:

**Theorem.** *Fix two arbitrary integers $q$ and $r$, and let*

$$D^{\mathrm{pr}}(q) \cap N = \{p_1, \ldots, p_k; c_1, \ldots, c_s\},$$
$$D^{\mathrm{pr}}(r) \cap N = \{p_1, \ldots, p_k; d_1, \ldots, d_t\}.$$

*Then there exists a unique strict isomorphism between $\mathbb{W}_N^q(A)$ and $\mathbb{W}_N^r(A)$ if and only if $A$ is a $\mathbb{Z}[\frac{1}{c_i}, \frac{1}{d_j}\colon 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t]$-algebra.*

The second problem is to establish isomorphisms of functors for coprime truncation sets $M$ and $N$, which can be stated as follows:

**Theorem.** *Let $q$ be any integer, and $M, N$ be truncation sets with $M \cap N = \{1\}$. Then we have isomorphisms of functors,*

$$\mathbb{W}_N^q \circ \mathbb{W}_M^q \cong \mathbb{W}_{MN}^q \quad \textit{and} \quad Nr_N^q \circ Nr_M^q \cong Nr_{MN}^q.$$

This paper is organized as follows. In Section 2, we review prerequisites on the ring of nested Witt vectors, the nested necklace ring, and their $q$-deformation. In Section 3, we study properties and relations among the objects introduced in Section 2. As a by-product, $N$-nested cyclotomic identities will be given (see Theorems 11 and 12). The main results will appear in Section 4. Many structural properties will be investigated for $q$-deformed nested Witt vectors and nested necklace rings. In the final section, we deal with applications which arise naturally in our frame. Generalized Möbius $\mu$-functions and certain polynomials taking integer values at integer arguments will be investigated.

## 2. Preliminaries

Unless otherwise stated the rings we consider will be commutative, but not necessarily unital. Denote by *Ring* the category of rings. In this section, we introduce several ring-valued covariant functors of interest to us.

Let $N \subseteq \mathbb{N}$ be a truncation set. The *$N$-nested ghost ring functor*, $\mathrm{gh}_N\colon Ring \to Ring$, associates to each object the ring whose underlying set equals $A^N$ and whose operations are defined componentwise, and to each morphism $f\colon A \to B$ associates the morphism

$\mathrm{gh}_N(f):(X_n)_{n\in N}\mapsto (f(X_n))_{n\in N}$. Under the above notation, the functor of *the ring of $N$-nested Witt vectors*, $\mathbb{W}_N: \mathit{Ring}\to \mathit{Ring}$, is characterized by the following three properties:

(i) As a set, $\mathbb{W}_N(A)$ equals $A^N$.
(ii) For any ring homomorphism $f:A\to B$, the map

$$\mathbb{W}_N(f):\mathbb{W}_N(A)\to\mathbb{W}_N(B),\qquad (X_n)_{n\in N}\mapsto \big(f(X_n)\big)_{n\in N}$$

is a ring homomorphism.
(iii) The map

$$\Phi_N:\mathbb{W}_N(A)\to \mathrm{gh}_N(A),\qquad (X_n)_{n\in N}\mapsto \left(\sum_{d|n} dX_d^{\frac{n}{d}}\right)_{n\in N}$$

is a ring homomorphism.

For more information refer to [14].

**Example 1.**

(a) Let $N$ be the set of divisors of $n$ for some $n\in\mathbb{N}$. Note that the subgroups of the multiplicative cyclic group $C(n)$ of order $n$ are parametrized naturally by their index in $C(n)$. It follows that the functor $\mathbb{W}_N$ is none other than the functor of Witt–Burnside rings associated to the group $C(n)$, denoted by $\mathbb{W}_{C(n)}$.
(b) Typical examples of infinite truncation sets are $\mathbb{N}$ and $\{1,p,p^2,\ldots\}$, where $p$ is any prime. In the literature, $\mathbb{W}_{\mathbb{N}}$, which is usually denoted by $\mathbb{W}$, is called the functor of *the ring of Witt vectors*. Similarly, $\mathbb{W}_{\{1,p,p^2,\ldots\}}$, which is usually denoted by $\mathbb{W}_p$, is called the functor of *the ring of $p$-typical Witt vectors*. As indicated in [3], $\mathbb{W}$ and $\mathbb{W}_p$ coincide with the functors of Witt–Burnside rings $\mathbb{W}_{\hat{C}}$ and $\mathbb{W}_{\hat{C}_p}$, respectively. Here, $\hat{C}$ represents the profinite completion of the infinite cyclic group $C$, and $\hat{C}_p$ the pro-$p$-completion of the infinite cyclic group $C$.

When $A$ is a commutative ring with identity, $\mathbb{W}_N(A)$ can be viewed as a subring of $\mathbb{W}(A)$ under the injection

$$\iota_W:\mathbb{W}_N(A)\to\mathbb{W}(A),\qquad (X_n)_{n\in N}\mapsto (\bar{X}_m)_{m\in\mathbb{N}},$$

where $\bar{X}_m$ is defined to be $X_m$ if $m\in N$, and 0 otherwise. So, in the category of commutative rings with identity, $\mathbb{W}_N$ can be viewed as a subfunctor of $\mathbb{W}$ under the above natural transformation. A remarkable property of $\mathbb{W}_N$ is that it has a $q$-deformation, where $q$ ranges over the set of integers. This phenomenon was first observed in [7,10] in case $N=\mathbb{N}$, and extended to the level of Witt–Burnside rings in [12]. According to this literature, there exists a unique functor $\mathbb{W}^q:\mathit{Ring}\to\mathit{Ring}$ for every integer $q$, satisfying the following conditions:

(i) As a set, $\mathbb{W}^q(A)$ equals $A^{\mathbb{N}}$.
(ii) For any ring homomorphism $f:A\to B$, the map

$$\mathbb{W}^q(f):(X_n)_{n\in\mathbb{N}}\mapsto \big(f(X_n)\big)_{n\in\mathbb{N}}$$

is a ring homomorphism.

(iii) The map

$$\Phi^q : \mathbb{W}^q(A) \to \mathrm{gh}(A), \qquad (X_n)_{n\in\mathbb{N}} \mapsto \left(\sum_{d|n} dq^{\frac{n}{d}-1} X_d^{\frac{n}{d}}\right)_{n\in\mathbb{N}}$$

is a ring homomorphism.

One can show that when $q = 1$, $\mathbb{W}^1$ coincides with $\mathbb{W}$, and when $q = 0$, $\mathbb{W}^0$ has quite a simple structure. Furthermore, it can be deduced from the condition (iii) that for each object $A$ the ring operations of $\mathbb{W}^q(A)$ are completely determined by the universal polynomials $\{\mathfrak{s}_n^q\colon n \in \mathbb{N}\}$, $\{\mathfrak{p}_n^q\colon n \in \mathbb{N}\}$, and $\{\iota_n^q\colon n \in \mathbb{N}\}$, which are provided by the following equations:

$$\begin{aligned}
\left(\mathfrak{s}_n^q\right)_{n\in\mathbb{N}} &= \left(\Phi^q\right)^{-1}\left(\Phi^q(\mathbf{X}) + \Phi^q(\mathbf{Y})\right),\\
\left(\mathfrak{p}_n^q\right)_{n\in\mathbb{N}} &= \left(\Phi^q\right)^{-1}\left(\Phi^q(\mathbf{X}) \cdot \Phi^q(\mathbf{Y})\right),\\
(0, 0, \ldots) &= \left(\Phi^q\right)^{-1}\left(\Phi^q(\mathbf{X}) + \Phi^q\left(\left(\iota_n^q\right)_{n\in\mathbb{N}}\right)\right),
\end{aligned}$$

respectively, where $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ and $\mathbf{Y} = (Y_n)_{n\in\mathbb{N}}$. Note that $X_n$'s and $Y_n$'s are regarded as indeterminates. The coefficients of these polynomials belong to $\mathbb{Q}[q]$. However, miraculously,

$$\begin{aligned}
\mathfrak{s}_n^q, \mathfrak{p}_n^q &\in \mathbb{Z}[X_d, Y_d\colon d \mid n],\\
\iota_n^q &\in \mathbb{Z}[X_d\colon d \mid n]
\end{aligned}$$

for every $n \in \mathbb{N}$ when we specialize $q$ to any integer. In particular, in view of the fact that $d$ ranges over the set of divisors of $n$, one can also define a modified functor, $\mathbb{W}_N^q : \mathit{Ring} \to \mathit{Ring}$, satisfying the following conditions:

(i) As a set, $\mathbb{W}_N^q(A)$ equals $A^N$.
(ii) For any ring homomorphism $f : A \to B$, the map $\mathbb{W}_N^q(f) : (X_n)_{n\in N} \mapsto (f(X_n))_{n\in N}$ is a ring homomorphism.
(iii) The map

$$\Phi_N^q : \mathbb{W}_N^q(A) \to \mathrm{gh}_N(A), \qquad (X_n)_{n\in N} \mapsto \left(\sum_{d|n} dq^{\frac{n}{d}-1} X_d^{\frac{n}{d}}\right)_{n\in N}$$

is a ring homomorphism.

For each object $A$, $\mathbb{W}^q(A)$ has a structure of a filtered ring. It has the *Verschiebung operator* $V_m$ for every positive integer $m$, which is defined by

$$V_m(\mathbf{X})(n) = \begin{cases} X_i & \text{if } n = mi,\\ 0 & \text{otherwise.}\end{cases} \tag{2.1}$$

We have a descending filtration on $\mathbb{W}^q(A)$ given by $V_m\mathbb{W}^q(A)$ for each $m \in \mathbb{N}$. It should be mentioned that $V_m\mathbb{W}^q(A)$ is closed under the ring operations of $\mathbb{W}^q(A)$, but the unit elements of $V_m\mathbb{W}^q(A)$ and $\mathbb{W}^q(A)$, if they exist, do not coincide.

In 1983, Metropolis and Rota [8] provided a combinatorial understanding of $\mathbb{W}(\mathbb{Z})$, the ring of Witt vectors over $\mathbb{Z}$, by using the combinatorics of necklaces. They constructed a functor, $Nr : Ring \to Ring$, satisfying the property that $\mathbb{W}(\mathbb{Z})$ is isomorphic to $Nr(\mathbb{Z})$. Compared with $\mathbb{W}(\mathbb{Z})$, the advantage of $Nr(\mathbb{Z})$ may be twofold as follows: First, it is very easy to describe. Second, it has a very nice interpretation. Indeed, $Nr(\mathbb{Z})$ can be realized as the Burnside–Grothendieck ring of almost finite cyclic sets (see [8]). Let us now restrict the index from $\mathbb{N}$ to $N$ to get the $N$-nested version of $Nr$. Denote this functor by $Nr_N$. On the category of commutative rings with identity, it can be viewed as a subfunctor of $Nr$ under the injective natural transformation

$$\iota_N : Nr_N(A) \to Nr(A), \qquad (X_n)_{n\in N} \mapsto (\bar{X}_m)_{m\in\mathbb{N}},$$

where $\bar{X}_m$ is defined to be $X_m$ if $m \in N$, and 0 otherwise.

$Nr$ has a $q$-deformation, denoted by $Nr^q : Ring \to Ring$, as $q$ ranges over the set of integers (see [10,12]). In exactly the same way as $\mathbb{W}^q_N$ has been constructed from $\mathbb{W}^q$, it can be shown that there exists a $q$-version of $Nr_N$. More precisely, for every integer $q$, one has a unique functor, $Nr^q_N : Ring \to Ring$, subject to the following conditions:

(i) As a set, it is $A^N$.
(ii) For any ring homomorphism $f : A \to B$, the map $Nr^q_N(f) : \mathbf{X} \mapsto (f(X_n))_{n\in N}$ is a ring homomorphism for $\mathbf{X} = (X_n)_{n\in N}$.
(iii) The map,

$$\varphi^q_N : Nr^q_N(A) \to \mathrm{gh}_N(A), \qquad \mathbf{X} \mapsto \left(\sum_{d|n} dq^{\frac{n}{d}-1}X_d\right)_{n\in N},$$

is a ring homomorphism.

We call $Nr^q_N$ the functor of *$q$-deformed $N$-nested necklace rings*. In particular, in case $N = \mathbb{N}$, the corresponding ring $Nr^q(A)$ has a structure of a filtered $A$-algebra associated with Verschiebung operators defined as in (2.1). Consequently we have a descending filtration on $Nr^q(A)$ given by the algebras $V_mNr^q(A)$ for $m \geqslant 1$. Note that the unit elements of $V_mNr^q(A)$ and $Nr^q(A)$, if they exist, do not coincide. Denote by $\mathbb{Z}_N := \mathbb{Z}[\frac{1}{n}\colon n \in N]$ the localization of $\mathbb{Z}$ by $N$.

**Lemma 2.** *Let $N$ be a truncation set and $q$ be any integer. Then we have the following characterizations.*

(a) *$\Phi^q_N(A)$ is injective $\Leftrightarrow$ $\varphi^q_N(A)$ is injective $\Leftrightarrow$ $A$ has no $N$-torsion.*
(b) *$\Phi^q_N(A)$ is surjective $\Leftrightarrow$ $\varphi^q_N(A)$ is surjective $\Leftrightarrow$ $\Phi^q_N(A)$ is bijective $\Leftrightarrow$ $\varphi^q_N(A)$ is bijective $\Leftrightarrow$ $A$ is a $\mathbb{Z}_N$-algebra.*

To state the relation between $\mathbb{W}_N$ and $Nr_N$ let us introduce $q$-necklace polynomials and binomial rings. For complete information see [12]. Given $m \in \mathbb{N}$, let us define a corresponding matrix, $\zeta_m^q$, on the lattice $D(m)$ of divisors of $m$, by

$$\zeta_m^q(d_1, d_2) = \begin{cases} q^{\frac{d_2}{d_1}-1} & \text{if } d_1 \mid d_2, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\mu_m^q$ be its inverse. Set

$$M^q(x, n) := \frac{1}{n} \sum_{d|n} \mu_n^q(d, n) q^{d-1} x^d. \tag{2.2}$$

These polynomials are called *$q$-necklace polynomials* since it counts the number of equivalence classes of aperiodic $q$-words of length $n$ out of $x$-letters in case $x$ is a positive integer. Considering $q$-necklace polynomials as function on a ring $A$, they do not make sense on every ring. Therefore, when dealing with $q$-necklace polynomials, we require the condition that $A$ should have a binomial ring structure. For the definition of binomial rings refer to [11, Section 2].

**Lemma 3.** [11,12,15]

(a) *Let $A$ be a torsion-free commutative ring with identity and such that $a^p = a \bmod pA$ if $p$ is a prime. Then $A$ has a unique special $\lambda$-ring structure with $\Psi^n = \mathrm{id}$ for all $n \in \mathbb{N}$.*
(b) *Let $A$ be a torsion-free commutative ring with identity. Then $A$ is a binomial ring if and only if $M^1(a, n) \in A$ for all $a \in A$ and $n \in \mathbb{N}$.*
(c) *If $A$ is a binomial ring, then $M^q(a, n) \in A$ for all $a \in A$ and $n, q \in \mathbb{N}$.*

From now on, we let $M(a, n) := M^1(a, n)$ for all $a \in A$ and $n \in \mathbb{N}$. By virtue of Lemma 3(c) one can define the map

$$M^q : A \to Nr^q(A), \qquad a \mapsto \big(M^q(a, 1), M^q(a, 2), \ldots\big)$$

if $A$ is a binomial ring. The following statement plays a fundamental role in the theory of $q$-deformed Witt vectors.

**Theorem 4.** [12] *Let $A$ be any binomial ring. Then, for any integer $q$, the map*

$$\tau^q : \mathbb{W}^q(A) \to Nr^q(A), \qquad (X_n)_{n \in \mathbb{N}} \mapsto \sum_{r \geqslant 1} V_r M^q(X_r),$$

*is a ring isomorphism. Furthermore, it satisfies the relation $\Phi^q = \varphi^q \circ \tau^q$.*

Observe that the $n$th component of $\tau^q((X_n)_{n \in \mathbb{N}})$ is given by $\sum_{d|n} M^q(X_d, \frac{n}{d})$. It follows that $\tau^q(\mathbb{W}_N^q(A)) = Nr_N^q(A)$. By restricting $\tau^q$ to $\mathbb{W}_N^q(A)$ one can derive a $N$-nested version of Theorem 4.

**Theorem 5.** *Let $A$ be any binomial ring. Then, for any integer $q$, the map*

$$\tau_N^q : \mathbb{W}_N^q(A) \to Nr_N^q(A), \qquad (X_n)_{n\in N} \mapsto \left(\sum_{d|n} M^q\left(X_d, \frac{n}{d}\right)\right)_{n\in N},$$

*is a ring isomorphism. Furthermore, it satisfies the relation $\Phi_N^q = \varphi_N^q \circ \tau_N^q$.*

## 3. Artin–Hasse-like exponential map and big diagram

In 1956, Grothendieck introduced a functor, denoted by $\Lambda$, from the category of unital commutative rings to the category of special λ-rings. As a set, $\Lambda(A)$ coincides with

$$1 + A[\![t]\!]^+ := \left\{1 + \sum_{n=1}^{\infty} x_n t^n : \ x_n \in A\right\}$$

for a commutative ring $A$ with identity. For its ring structure we refer to [11, Section 2], where the notation $\Lambda_1$ has been used to denote $\Lambda$. Viewed as a functor from the category of unital commutative rings to the category of special λ-rings, $\Lambda$ is isomorphic to $\mathbb{W}$ under Artin–Hasse-like exponential map

$$\mathcal{E} : \mathbb{W}(A) \to \Lambda(A), \qquad (x_n)_{n\in\mathbb{N}} \mapsto \prod_{n\in\mathbb{N}} \frac{1}{1 - x_n t^n}.$$

The structure of $\mathbb{W}(\mathbb{Z})$ was extensively studied by Dress and Siebeneicher [4]. They provided a diagram which illustrates close connections between the ring of Witt vectors (over $\mathbb{Z}$) and necklace ring, ghost ring, and Grothendieck ring of formal power series with constant term 1 (over $\mathbb{Z}$).

The purpose of this section is to derive a $N$-nested version of Artin–Hasse-like exponential map and to provide an analogous global picture illustrating the structure of the ring of $N$-nested Witt vectors very nicely. To do this we need to recall the following lemma on the symmetric map.

**Lemma 6.** [11] *Let $A$ be a binomial ring. The symmetric map,*

$$s_t : Nr(A) \to \Lambda(A), \qquad (b_n)_{n\in\mathbb{N}} \mapsto \prod_{n\in\mathbb{N}} \left(\frac{1}{1 - t^n}\right)^{b_n},$$

*is a ring isomorphism. Furthermore, it satisfies the relation that $\mathcal{E} = s_t \circ \tau$.*

Let $A$ be a binomial ring. From Theorem 4 and Lemma 6 it is obvious that the following diagram

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "\tau(\cong)"] \arrow[d, "\Phi"] & Nr(A) \arrow[r, "s_t(\cong)"] \arrow[d, "\varphi"] & \Lambda(A) \arrow[d, "t\frac{d}{dt}\log"] \\
\mathrm{gh}(A) \arrow[r, "\mathrm{id}"] & \mathrm{gh}(A) \arrow[r, "\text{identification}"] & tA[\![t]\!]
\end{tikzcd} \tag{3.1}$$

is commutative. Here, the map *identification*: $\mathrm{gh}(A) \to tA[\![t]\!]$ sends $(x_n)_{n\in\mathbb{N}}$ to the power series $\sum_{n\in\mathbb{N}} x_n t^n$.

Denote by $N^\perp$ the set $\{m \in \mathbb{N}\colon\ \gcd(m,n) = 1 \text{ for all } n \in N\}$, where $N$ is a truncation set. By definition we have $N \cap N^\perp = \{1\}$. Now we introduce a truncation set of a very simple form. A truncation set $N$ is called *monoidal* if it is generated by a set of prime numbers. If $N$ is monoidal, then $N^\perp$ is uniquely determined by the two identities $N \cap N^\perp = \{1\}$ and $NN^\perp = \mathbb{N}$. More generally the following statement holds:

**Lemma 7.** *Let $N$ be a truncation set. Then $NN^\perp = \mathbb{N}$ if and only if $N$ is monoidal.*

With this notation, we define a map, which is injective,

$$\eta_G : \mathrm{gh}_N(A) \to \mathrm{gh}(A), \qquad (X_n)_{n\in N} \mapsto (\bar{X}_m)_{m\in\mathbb{N}},$$

where

$$\bar{X}_m := \begin{cases} X_m & \text{if } m \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A$ be unital. Note that $\eta_G$ does not preserve identity elements unless $N$ equals $\mathbb{N}$. Let us consider a map $\eta_N : Nr_N(A) \to Nr(A)$ so that the diagram

$$\begin{tikzcd} Nr_N(A) \arrow[r, "\eta_N"] \arrow[d, "\varphi_N"'] & Nr(A) \arrow[d, "\varphi"'] \\ \mathrm{gh}_N(A) \arrow[r, "\eta_G"] & \mathrm{gh}(A) \end{tikzcd} \tag{3.2}$$

be commutative. Letting $(\breve{a}_m)_{m\in\mathbb{N}}$ be the image of $(a_n)_{n\in N}$ by $\eta_N$, the commutativity condition says that for every $n \in \mathbb{N}$

$$\sum_{d|n} d\breve{a}_n = \begin{cases} \sum_{d|n} da_d & \text{if } n \in N, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Applying Möbius inversion formula to Eq. (3.3) we obtain that

$$\breve{a}_n = \frac{1}{n} \sum_{\substack{d|n \\ d\in N}} \mu\left(\frac{n}{d}\right)\left(\sum_{e|d} ea_e\right). \tag{3.4}$$

**Example 8.**

(a) $\breve{a}_n = a_n$ for every $n$ if $N = \mathbb{N}$, and

$$\breve{a}_n = \frac{\mu(n_2)}{n_2} a_{n_1}$$

if $N$ is monoidal and $n = n_1 n_2$ with $n_1 \in N$, $n_2 \in N^\perp$.

(b) If $a_n = M(x,n)$, then $\breve{a}_n$ reduces to

$$\frac{1}{n}\sum_{\substack{d|n\\ d\in N}} \mu\left(\frac{n}{d}\right)x^d.$$

Indeed, it can be regarded as a truncated version of the polynomial $M(x,n)$. So, we will denote it by $M_N(x,n)$ (compare with Section 5.1).

Unless otherwise stated, we assume that $N$ is monoidal in the remaining part of this section. Let $A$ be a $\mathbb{Z}_{N^\perp}$-algebra and define $\eta_N : Nr_N(A) \to Nr(A)$ by

$$\breve{a}_n = \frac{\mu(n_2)}{n_2} a_{n_1}$$

for $n = n_1 n_2$ with $n_1 \in N$, $n_2 \in N^\perp$. Note that $\eta_N$ is well defined by the condition of $A$. Also $A$ has a binomial ring structure, which follows from Lemma 3(a). Therefore, we obtain the commutative diagram

$$\begin{tikzcd}
\mathbb{W}_N(A) \arrow[r, "\tau_N(\cong)"] \arrow[d, "\Phi_N"] & Nr_N(A) \arrow[r, "\eta_N"] \arrow[d, "\varphi_N"] & Nr(A) \arrow[r, "s_t(\cong)"] \arrow[d, "\varphi"] & \Lambda(A) \arrow[d, "t\frac{d}{dt}\log"] \\
\mathrm{gh}_N(A) \arrow[r, "\mathrm{id}"] & \mathrm{gh}_N(A) \arrow[r, "\eta_G"] & \mathrm{gh}(A) \arrow[r, "\text{identification}"] & tA[\![t]\!]
\end{tikzcd} \tag{3.5}$$

by combining diagram (3.1) with diagram (3.2). Define $\Lambda_N(A)$ by the image of $\mathbb{W}_N(A)$ for $s_t \circ \eta_N \circ \tau_N$. Note that it is not a subring of $\Lambda(A)$ because the unit element of $\Lambda(A)$ and $\Lambda_N(A)$ do not coincide. However, since $s_t$, $\eta_N$, and $\tau_N$ are all injective, we can obtain a ring isomorphism $s_t \circ \eta_N \circ \tau_N : \mathbb{W}_N(A) \to \Lambda_N(A)$ by the restriction of the range. Let

$$E_N(t) := \prod_{d\in N^\perp}\left(\frac{1}{1-t^d}\right)^{\frac{\mu(d)}{d}}.$$

As in [14], let us define a ring isomorphism

$$\mathcal{E}_N : \mathbb{W}_N(A) \to \Lambda_N(A), \qquad (x_n)_{n\in N} \mapsto \prod_{c\in N} E_N\left(x_c t^c\right).$$

**Theorem 9.** *Let $N$ be a monoidal truncation set and $A$ a $\mathbb{Z}_{N^\perp}$-algebra. The following statements hold*:

(a) *$\mathcal{E}_N$ is a ring isomorphism, and $\mathcal{E}_N = s_t \circ \eta_N \circ \tau_N$.*
(b) *If $A$ is of characteristic zero, then $\Lambda_N(A)$ equals*

$$\Xi_N := \left\{ f(t) \in \Lambda(A)\colon\ \log f(t) \text{ is of the form } \sum_{n\in N} a_n t^n,\ \text{where } a_n \in A\otimes\mathbb{Q}\right\}.$$

(c) *The power series*

$$\prod_{n\in N^{\perp}}\left(\frac{1}{1-t^n}\right)^{\frac{\mu(n)}{n}}$$

*is in $\Lambda_N(A)$, and the identity of $\Lambda_N(A)$.*

**Proof.** (a) Since $t\frac{d}{dt}\log$ is injective, it suffices to show that

$$t\frac{d}{dt}\log\circ\mathcal{E}_N(\mathbf{X})=t\frac{d}{dt}\log\circ s_t\circ\eta_N\circ\tau_N(\mathbf{X})$$

for all $\mathbf{X}=(x_n)_{n\in N}\in\mathbb{W}_N(A)$. Note that

$$t\frac{d}{dt}\log\circ\mathcal{E}_N(\mathbf{X})=\sum_{c\in N}c\sum_{n\in N}x_c^n t^{cn}=\sum_{n\in N}\left(\sum_{d|n}dx_d^{\frac{n}{d}}\right)t^n.$$

From the commutativity of diagram (3.5) it follows that

$$t\frac{d}{dt}\log\circ s_t\circ\eta_N\circ\tau_N(\mathbf{X})=\sum_{n\in N}\left(\sum_{d|n}dx_d^{\frac{n}{d}}\right)t^n.$$

So, we are done.

(b) If $f(t)\in\Lambda_N(A)$, then

$$t\frac{d}{dt}\log f(t)=\sum_{n\in N}\left(\sum_{d|n}dx_d^{\frac{n}{d}}\right)t^n$$

for some $(x_n)_{n\in N}\in\mathbb{W}_N(A)$. This follows from the commutativity of diagram (3.5). Thus, we have

$$\log f(t)=\sum_{n\in N}\frac{1}{n}\left(\sum_{d|n}dx_d^{\frac{n}{d}}\right)t^n\in\Xi_N.$$

For the converse, let us assume that $f(t)\in\Xi_N$. In view of Theorem 9(a) together with the fact that $t\frac{d}{dt}\log f(t)$ is in the image of identification $\circ\,\eta_G\circ\Phi_N$ over $A\otimes\mathbb{Q}$, one can write $f(t)$ as $\prod_{c\in N}E_N(x_ct^c)$ for some $(x_n)_{n\in N}\in\mathbb{W}_N(A\otimes\mathbb{Q})$. Since $f(t)\in\Lambda(A)\cap\Lambda_N(A\otimes\mathbb{Q})$ the desired result follows.

(c) By definition

$$\eta_N\circ\tau_N(1,0,0,\ldots)=\begin{cases}\frac{\mu(n)}{n} & \text{if } n\in N^{\perp},\\ 0 & \text{otherwise.}\end{cases}$$

It follows that

$$\mathcal{E}_N(1,0,0,\ldots)=\prod_{n\in N^{\perp}}\left(\frac{1}{1-t^n}\right)^{\frac{\mu(n)}{n}}.$$

This implies the desired result immediately. □

Letting $s_{t,N} := s_t \circ \eta_N$, we have a ring isomorphism

$$s_{t,N} : Nr_N(A) \to \Lambda_N(A), \qquad (x_n)_{n\in N} \mapsto \prod_{\substack{c\in N\\ d\in N^{\perp}}} \left(\frac{1}{1-t^{cd}}\right)^{\frac{\mu(d)}{d}x_c}.$$

By the commutativity of diagram (3.5) the identity

$$\prod_{c\in N} E_N\big(t^c\big)^{x_c} = \exp\biggl(\sum_{n\in N}\frac{1}{n}\sum_{d|n} d x_d\, t^n\biggr) \tag{3.6}$$

is straightforward. Particularly, if $(x_n)_{n\in N}$ is the identity $(1,0,\dots,0)$, then Eq. (3.6) reduces to the formula:

$$E_N(t) = \exp\biggl(\sum_{n\in N}\frac{1}{n}t^n\biggr).$$

Next, we would like to introduce another injective map instead of $\eta_G$. Consider an injective ring homomorphism

$$\iota_G : \mathrm{gh}_N(A) \to \mathrm{gh}(A), \qquad (a_n)_{n\in N} \mapsto (\tilde{a}_m)_{m\in\mathbb{N}},$$

where

$$\tilde{a}_m := a_{m_1}$$

for $m = m_1 m_2$ with $m_1 \in N$, $m_2 \in N^{\perp}$. It is not difficult to show that the following diagrams

$$\begin{array}{ccc} \mathbb{W}_N(A) & \xrightarrow{\iota_W} & \mathbb{W}(A) \\ {\scriptstyle\Phi_N}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathrm{gh}_N(A) & \xrightarrow{\iota_G} & \mathrm{gh}(A), \end{array} \qquad \begin{array}{ccc} Nr_N(A) & \xrightarrow{\iota_N} & Nr(A) \\ {\scriptstyle\varphi_N}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathrm{gh}_N(A) & \xrightarrow{\iota_G} & \mathrm{gh}(A), \end{array} \qquad \begin{array}{ccc} \mathbb{W}_N(A) & \xrightarrow{\iota_W} & \mathbb{W}(A) \\ {\scriptstyle\tau_N}\downarrow & & \downarrow{\scriptstyle\tau} \\ Nr_N(A) & \xrightarrow{\iota_N} & Nr(A) \end{array}$$

are all commutative. Particularly, by combining the second diagram with (3.1) we can derive the following commutative diagram

$$\begin{array}{ccccccc} \mathbb{W}_N(A) & \xrightarrow{\tau_N(\cong)} & Nr_N(A) & \xrightarrow{\iota_N} & Nr(A) & \xrightarrow{s_t(\cong)} & \Lambda(A) \\ \downarrow{\scriptstyle\Phi_N} & & \downarrow{\scriptstyle\varphi_N} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle t\frac{d}{dt}\log} \\ \mathrm{gh}_N(A) & \xrightarrow{\mathrm{id}} & \mathrm{gh}_N(A) & \xrightarrow{\iota_G} & \mathrm{gh}(A) & \xrightarrow{\text{identification}} & tA[\![t]\!] \end{array} \tag{3.7}$$

if $A$ is a binomial ring.

**Theorem 10.** *Let $N$ be a monoidal truncation set and $A$ be a binomial ring. Then it holds that*

$$\prod_{n\in N}\left(\frac{1}{1-t^n}\right)^{x_n}=\exp\left(\sum_{\substack{c\in N\\ d\in N^{\perp}}}\left(\frac{1}{cd}\sum_{e|c}ex_e\right)t^{cd}\right)$$

*for every $(x_n)_{n\in N}\in Nr_N(A)$.*

**Proof.** This follows from the relation that $t\frac{d}{dt}\log\circ s_t\circ\iota_N=\text{identification}\circ\iota_G\circ\varphi_N$. $\square$

We close this section with two applications associated with diagrams (3.5) and (3.7). The first application is so-called *$N$-nested cyclotomic identities*. For full information of the cyclotomic identity see [4,8,9].

**Theorem 11.** *Let $N$ be a monoidal truncation set and let $x, t$ be indeterminates. Then the following relations hold*:

(a) $$E_N(xt)=\prod_{c\in N}E_N\left(t^c\right)^{M(x,c)}.$$

(b) $$\prod_{n\in N}\left(\frac{1}{1-t^n}\right)^{M(x,n)}=\exp\left(\sum_{\substack{c\in N\\ d\in N^{\perp}}}\frac{1}{cd}\left(xt^d\right)^c\right).$$

**Proof.** (a) The identity to prove is nothing but the re-statement of the relation

$$\begin{aligned}\mathcal{E}_N(x,0,0,\ldots)&=s_t\circ\eta_N\circ\tau_N(x,0,0,\ldots)\\ &=\left(t\frac{d}{dt}\log\right)^{-1}\circ\text{identification}\circ\eta_G\circ\Phi_N(x,0,0,\ldots).\end{aligned}$$

(b) Similarly, the identity to prove is the re-statement of the relation

$$s_t\circ\iota_N\circ\tau_N(x,0,0,\ldots)=\left(t\frac{d}{dt}\log\right)^{-1}\circ\text{identification}\circ\iota_G\circ\Phi_N(x,0,0,\ldots).\qquad\square$$

**Theorem 12.** *Let $N$ be any truncation set and let $x, t$ be indeterminates. Then we have*

$$\prod_{n\in\mathbb{N}}\left(\frac{1}{1-t^n}\right)^{M_N(x,n)}=\exp\left(\sum_{n\in N}\frac{x^n}{n}t^n\right).$$

**Proof.** Define $\eta_N$ as in Eq. (3.4), and then apply the commutativity of diagram (3.5) to get the desired result. $\square$

Letting $N=\mathbb{N}$, we can recover the classical cyclotomic identity

$$\frac{1}{1-xt}=\prod_{n\in\mathbb{N}}\left(\frac{1}{1-t^n}\right)^{M(x,n)}.$$

Finally, we would like to remark the relations among certain power series associated with arithmetic functions. Let $f$ be an arithmetic function, and let $F(n) = \sum_{d|n} f(d)$. By Möbius inversion formula, we have

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

In our framework, the relation between $f$ and $F$ can be re-interpreted as

$$\varphi\left(\left(\frac{f(n)}{n}\right)_{n\in\mathbb{N}}\right) = \big(F(n)\big)_{n\in\mathbb{N}}.$$

Hence, from the relation

$$t\frac{d}{dt}\log\circ s_t = \text{identification}\circ\varphi,$$

it follows that

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{\frac{f(n)}{n}} = \exp\left(\sum_{n=1}^{\infty}\frac{F(n)}{n}t^n\right).$$

This formula is quite useful in case where the value of each $F(n)$ is explicitly known. Let us give some examples. First, let $f$ be the Möbius inversion function $\mu$. Then,

$$F(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and hence the resulting formula looks like

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{\frac{\mu(n)}{n}} = \exp(t).$$

If $f$ is the Euler's totient function $\phi$, we have

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{\frac{\phi(n)}{n}} = \exp\left(\frac{t}{1-t}\right).$$

Another noteworthy example is related with Liouville's function $\lambda$ which is defined by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of primes dividing $n$, counting multiplicities. In this case, we have

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{\frac{\lambda(n)}{n}} = \exp\left(\sum_{n=1}^{\infty}\frac{t^{n^2}}{n^2}\right).$$

## 4. $q$-Deformation of nested Witt vectors and nested necklace rings

### 4.1. Deformed nested Witt vectors

Let $A$ be a commutative ring with identity, and let $N$ be a truncation set. And then, we fix two arbitrary integers $q$ and $r$. The aim of this section is to provide a necessary and sufficient condition of $A$ so that $\mathbb{W}_N^q(A)$ and $\mathbb{W}_N^r(A)$ should be strictly isomorphic.

To begin with, we introduce the prerequisite notations. Given a integer $q$, we define $D(q)$ by the set of divisors of $q$, and $D^{\mathrm{pr}}(q)$ by the set of prime divisors of $q$, respectively. Conventionally, $D(0)$ will denote the set of positive integers $\mathbb{N}$, and $D^{\mathrm{pr}}(0)$ the set of all primes in $\mathbb{N}$. So, $D^{\mathrm{pr}}(0) \cap N$ will represent the set of all primes belonging to $N$.

**Definition 13.** Let $q$ and $r$ be arbitrary integers. Under the above notation, $\mathbb{W}_N^q(A)$ is said to be *strictly isomorphic* to $\mathbb{W}_N^r(A)$ if there exists a ring isomorphism, say $\varpi_q^r : \mathbb{W}_N^q(A) \to \mathbb{W}_N^r(A)$, satisfying $\Phi_N^q = \Phi_N^r \circ \varpi_q^r$. In this case, $\varpi_q^r$ is called a *strict isomorphism*.

The main result of this section is to prove the following criterion of classification:

**Theorem 14.** *Fix two arbitrary integers $q$ and $r$, and let*

$$D^{\mathrm{pr}}(q) \cap N = \{p_1, \ldots, p_k; c_1, \ldots, c_s\},$$
$$D^{\mathrm{pr}}(r) \cap N = \{p_1, \ldots, p_k; d_1, \ldots, d_t\}.$$

*That is, $p_i$'s are prime divisors in $D^{\mathrm{pr}}(q) \cap D^{\mathrm{pr}}(r) \cap N$. Then, there exists a unique strict isomorphism between $\mathbb{W}_N^q(A)$ and $\mathbb{W}_N^r(A)$ if and only if $A$ is a $\mathbb{Z}[\frac{1}{c_i}, \frac{1}{d_j}\colon 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t]$-algebra.*

In view of Theorem 14, it is immediate that the functors $\mathbb{W}_N^q$ and $\mathbb{W}_N^r$ are naturally isomorphic if there are no $c_i$'s and $d_j$'s. In proving this assertion, the following lemma plays a crucial role. For the content on formal group laws refer to [5].

**Lemma 15.** [13] *For any integers $q$ and $r$, let us consider the corresponding formal group laws $F_q(X, Y) = X + Y - qXY$ and $F_r(X, Y) = X + Y - rXY$, respectively. Let*

$$D^{\mathrm{pr}}(q) = \{p_1, \ldots, p_k; e_1, \ldots, e_s\},$$
$$D^{\mathrm{pr}}(r) = \{p_1, \ldots, p_k; f_1, \ldots, f_t\}.$$

*Then, we have*

(a) *$F_q(X, Y)$ is strictly isomorphic to $F_r(X, Y)$ over the ring*

$$\mathbb{Z}\left[\frac{1}{e_i}, \frac{1}{f_j}\colon 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t\right].$$

(b) *If $q$ and $r$ have the same set of prime divisors, then $F_q(X, Y)$ is strictly isomorphic to $F_r(X, Y)$ over $\mathbb{Z}$.*

Now, let us consider the set of identities coming from the equation

$$\Phi_N^q(\mathbf{X}) = \Phi_N^r(\mathbf{Y}),$$

equivalently

$$\sum_{d|n} dq^{\frac{n}{d}-1} X_d^{\frac{n}{d}} = \sum_{d|n} dr^{\frac{n}{d}-1} Y_d^{\frac{n}{d}} \quad \text{for all } n \in N. \tag{4.1}$$

Here, $\mathbf{X}$ represents the vector $(X_n)_{n\in N}$ and $\mathbf{Y}$ the vector $(Y_n)_{n\in N}$, respectively. Also, we assume that the coefficients are defined over $\mathbb{Q}$. Using Eq. (4.1), let us express $Y_n$, $n \in N$, as a polynomial in $X_d$, $d \mid n$, which can be done in an inductive way. Then, one can easily show that

$$Y_n - X_n \in \mathbb{Z}_N[X_d\colon\ d \mid n \text{ and } d < n].$$

On the other hand, it follows from Lemma 15 that

$$Y_n - X_n \in \mathbb{Z}\left[\frac{1}{e_i}, \frac{1}{f_j}\colon\ 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t\right][X_d\colon\ d \mid n].$$

This is because every formal group law $F_q(X, Y)$, $q \in \mathbb{Z}$, induces functorially the corresponding ring of Witt vectors $\mathbb{W}^q(A)$ so that $\Phi^q$ is a ring homomorphism (see [5]). Putting these two results together, we can conclude that the coefficients of $Y_n$ are defined on the ring

$$\mathbb{Z}\left[\frac{1}{e_i}, \frac{1}{f_j}\colon\ 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t \text{ and } e_i, f_i \in N\right]. \tag{4.2}$$

Denote the polynomial obtained from $Y_n$ by $\mathcal{P}_n(X_d\colon\ d \mid n)$. Note that $\mathcal{P}_n$ is of the form

$$X_n + \text{a polynomial in } X_d\text{'s}, \quad d \mid n \text{ and } d \neq n. \tag{4.3}$$

Now, we are ready to prove our statement on classification.

**Proof of Theorem 14.** First, assume that $A$ is a $\mathbb{Z}[\frac{1}{c_i}, \frac{1}{d_j}\colon\ 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t]$-algebra. Equivalently, all the $c_i \cdot 1$'s and $d_j \cdot 1$'s are units in $A$. Using the universal polynomials $\{\mathcal{P}_n\colon\ n \in N\}$, let us define a map

$$\varpi_q^r : \mathbb{W}_N^q(A) \to \mathbb{W}_N^r(A), \qquad (a_n)_{n\in N} \mapsto \big(\mathcal{P}_n(a_d\colon\ d \mid n)\big)_{n\in N}.$$

Obviously, this map is well defined in view of Eq. (4.2). To show that $\varpi_q^r$ is bijective, observe $\mathcal{P}_1(X_1) = X_1$. Now, from Eq. (4.3) it follows that if $X_k$'s, $k < n$, are expressed in $\mathcal{P}_i$'s with $i \in N$ and $i \leqslant k$, then $X_n$ can be written as

$$\mathcal{P}_n + \frac{1}{n} \sum_{\substack{d|n \\ d\neq n}} d\big(r^{\frac{n}{d}-1}\mathcal{P}_d^{\frac{n}{d}} - q^{\frac{n}{d}-1} X_d^{\frac{n}{d}}\big).$$

This shows the surjectivity. The injectivity follows from the fact that the coefficient of $X_n$ in the polynomial $\mathcal{P}_n$ is 1 for all $n \in N$. In order to show that $\varpi_q^r$ is a ring homomorphism, let us observe the systems of equations, which are defined over $\mathbb{Q}$,

$$\begin{aligned}\varpi_q^r(\mathbf{X}+\mathbf{X}') &= \left(\Phi_N^r\right)^{-1} \circ \Phi_N^q(\mathbf{X}+\mathbf{X}') \\ &= \left(\Phi_N^r\right)^{-1} \circ \Phi_N^q(\mathbf{X}) + \left(\Phi_N^r\right)^{-1} \circ \Phi_N^q(\mathbf{X}').\end{aligned}$$

Equivalently,

$$\mathcal{P}_n(\mathfrak{s}_d\colon d \mid n) = \mathcal{P}_n(X_d\colon d \mid n) + \mathcal{P}_n\left(X'_d\colon d \mid n\right), \quad n \in N. \tag{4.4}$$

Here, $\mathfrak{s}_n = \mathfrak{s}_n(X_d, X'_d\colon d \mid n)$'s represent the polynomials determining the addition of the ring of nested Witt vectors. Note that polynomial-identities of Eq. (4.4) are defined over the ring

$$\mathbb{Z}\left[\frac{1}{c_i}, \frac{1}{d_j}\colon 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t\right].$$

This implies that $\varpi_q^r$ is additive. In the same manner one can show that $\varpi_q^r$ is also multiplicative. Finally, the fact that $\varpi_q^r$ is a strict isomorphism follows from its definition since $\varpi_q^r(\mathbf{X}) = \mathbf{Y}$. Furthermore, $\varpi_q^r$ is completely determined by the set $\{\mathcal{P}_n\colon n \in N\}$. It means that there can exist at most one strict isomorphism.

Conversely, let us assume that there exists a unique strict isomorphism, say $\varpi_q^r : \mathbb{W}^q(A) \to \mathbb{W}^r(A)$. Hence, we have $\Phi^q = \Phi^r \circ \varpi_q^r$. Assume that there exists a prime $p \in N$ which divides $q$ but not $r$. Letting

$$(a_n)_{n \in N} := \varpi_q^r(1, 0, 0, \ldots),$$

we have

$$q^{n-1} = \sum_{d \mid n} d r^{\frac{n}{d}-1} a_d^{\frac{n}{d}}, \quad n \geqslant 1.$$

In particular, $a_1 = 1$ and

$$r^{p-1} + p a_p = q^{p-1}. \tag{4.5}$$

Since $(p, r) = 1$, the *Little theorem of Fermat* says that $r^{p-1} = 1 + p\alpha$ for some $\alpha \in \mathbb{Z}$. Thus Eq. (4.5) can be written as

$$p\left(\alpha + a_p - \frac{q^{p-1}}{p}\right) = -1,$$

which means that $p \cdot 1$ is a unit in $A$. In the same manner, for all the primes $p$ in $N$ dividing $r$ but not $q$, $p \cdot 1$ also should be units in $A$. This proves our assertion. $\square$

**Corollary 16.** *Let $q$ and $r$ be arbitrary integers, and $A$ be a commutative ring with identity. Then, we have*

(a) *$\mathbb{W}^q(A)$ and $\mathbb{W}^r(A)$ are isomorphic as filtered rings associated with the operators $V_m$, $m \geqslant 1$ if they are strictly isomorphic.*
(b) *Assume that $A$ is an integral domain of characteristic zero. Then, $\mathbb{W}^q(A)$ is strictly isomorphic to $\mathbb{W}^r(A)$ if they are isomorphic as filtered rings associated with the operators $V_m$, $m \geqslant 1$.*

**Proof.** (a) Let us assume that we have a strict isomorphism $\varpi_q^r : \mathbb{W}^q(A) \to \mathbb{W}^r(A)$. For our purpose it suffices to show that $V_m \circ \varpi_q^r = \varpi_q^r \circ V_m$ for all $m \geqslant 1$. Observe that, over $\mathbb{Q}$, this equality is the case since $\Phi^q$ and $\Phi^r$ preserve the Verschiebung operators (see [7,10,12]) and they are also isomorphisms. Furthermore, it is given by a set of polynomial identities with all coefficients defined on the ring

$$\mathbb{Z}\left[\frac{1}{c_i}, \frac{1}{d_j}\colon\ 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t\right]$$

(see Theorem 14). Hence, over $\mathbb{Z}[\frac{1}{c_i}, \frac{1}{d_j}\colon\ 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t]$-algebras, it is the case that $V_m \circ \varpi_q^r = \varpi_q^r \circ V_m$. This gives the desired result.

(b) Let us assume that there is such an isomorphism $h : \mathbb{W}^q(A) \to \mathbb{W}^r(A)$. Extend this map to $h : \mathbb{W}^q(A \otimes \mathbb{Q}) \to \mathbb{W}^r(A \otimes \mathbb{Q})$ in the obvious way. Letting $\tilde{h} = \Phi^r \circ h \circ (\Phi^q)^{-1}$, $\tilde{h}$ is a filtered isomorphism on $\mathrm{gh}(A \otimes \mathbb{Q})$. For the vector $e_k = (0, 0, \ldots, \overset{k\text{th}}{1}, 0, 0, \ldots)$, $k \geqslant 1$, we have $\tilde{h}(e_k) \cdot \tilde{h}(e_k) = \tilde{h}(e_k)$. It follows that the coordinates of $\tilde{h}(e_k)$ are 0 or 1. Furthermore, from $\tilde{h}(e_k + e_l) \cdot \tilde{h}(e_k + e_l) = \tilde{h}(e_k + e_l)$, $k \neq l$, it follows that $\tilde{h}(e_k)\tilde{h}(e_l) = 0$. This equality implies that $\tilde{h}(e_k)$ and $\tilde{h}(e_l)$ cannot contain a 1 in the same column. Combining this observation with the fact that $\tilde{h}$ is a filtered isomorphism we can conclude that $\tilde{h}$ is the identity map. Consequently, $\Phi^r \circ h = \Phi^q$. Finally, by restricting the domain $A \otimes \mathbb{Q}$ to $A$, we can establish the desired result. □

As an application, let us apply Theorem 14 to some typical cases. To do this we need to introduce some auxiliary notations. First, we denote by $\mathbb{Z}_{(q),N}$ by the $N$-nested ring of integers localized at $q \neq 0$, that is, $\{m/n \in \mathbb{Q}\colon\ (n, q) = 1,\ n \in N\}$. Second, for a nonzero integer $q$, we mean by $\mathbb{Z}_{q \cap N}$ the ring

$$\mathbb{Z}\left[\frac{1}{p}\colon\ p \text{ is a prime in } N \text{ dividing } q\right].$$

**Corollary 17.** *Let $A$ be a commutative ring with identity. Then, the following statements hold*:

(a) *Let $q$ be a nonzero integer. Then, $A$ is a $\mathbb{Z}_{(q),N}$-algebra if and only if there exists a unique strict isomorphism, say $\varpi_0^q : \mathbb{W}_N^0(A) \to \mathbb{W}_N^q(A)$.*
(b) *Let $q$ be a nonzero integer. Then, $A$ is a $\mathbb{Z}_{q \cap N}$-algebra if and only if there exists a unique strict isomorphism, say $\varpi_1^q : \mathbb{W}_N(A) \to \mathbb{W}_N^q(A)$.*
(c) *Let $q$ vary over the set of integers. Then, $\mathbb{W}_N^q(\mathbb{Z})$ are classified up to strict isomorphism by the set of prime divisors of $q$ in $N$.*
(d) *For arbitrary integers $q$ and $r$, $\mathbb{W}_N^q(A)$ is strictly isomorphic to $\mathbb{W}_N^r(A)$ if $A$ is a $\mathbb{Z}_N$-algebra.*

(e) *If $q \cdot 1$ is not zero in $A$, that is, if $A$ has no $q$-torsion, then let us denote by $A_{q \cap N}$ the ring*

$$A\left[\frac{1}{p}\colon\ p \text{ is a prime divisor of } q \text{ belonging to } N\right].$$

*Then, the map, $\varpi_q^r : \mathbb{W}_N^q(B) \to \mathbb{W}_N^r(B)$, is a strict isomorphism where $B$ is $A_{q \cap N}$, or $A[\frac{1}{q}]$.*

**Proof.** (a) Let $q \neq 0$ and $r = 0$. Recall that $D^{\mathrm{pr}}(q) \cap N$ represents the set of all prime divisors of $q$ belonging to $N$ and $D^{\mathrm{pr}}(0) \cap N$ the set of all primes belonging to $N$, respectively. Hence there are no $c_i$'s and $d_j$'s are primes belonging to $N$ which do not divide $q$ (for the definition of $c_i$'s and $d_j$ see Theorem 14). Now, the desired result follows from Theorem 14.

(b) Let $r = 1$. Then, $D^{\mathrm{pr}}(1) \cap N$ is the empty set. Hence, in the context of Theorem 14, $c_j$'s are primes in $N$ which divides $q$, and no $d_j$'s exist. In the same way as in (a), we obtain the desired result.

(c) Note that the units in $\mathbb{Z}$ are only 1 and $-1$. Therefore $c_i$'s and $d_j$'s are nonunits. By Theorem 14, $\mathbb{W}_N^q(\mathbb{Z})$ and $\mathbb{W}_N^r(\mathbb{Z})$ are strictly isomorphic if and only if there are no $c_i$'s and $d_j$'s. On the other hand, there are no $c_i$'s and $d_j$'s if and only if $D^{\mathrm{pr}}(q) \cap N = D^{\mathrm{pr}}(r) \cap N$. So, we are done.

(d) By assumption, note that $p \cdot 1$ is a unit in $A$ for every prime $p$ belonging to $N$. Now, apply Theorem 14 to get the desired result.

(e) Observe that $A_{q \cap N}$ and $A[\frac{1}{q}]$ are $\mathbb{Z}_{q \cap N}$-algebras. Now, apply (b) to get the desired result. $\square$

Following the same way as in Theorem 14 we can also establish the following fact:

**Theorem 18.** *Let $q$ be any integer. Then the following statements are equivalent.*

(a) *$A$ is a $\mathbb{Z}_N$-algebra.*
(b) *The map $\Phi_N^q : \mathbb{W}_N^q(A) \to \mathrm{gh}_N(A)$ is a ring isomorphism.*

**Proof.** (a) $\Rightarrow$ (b): This part is almost straightforward (refer to [14, Lemma 1]).

(b) $\Rightarrow$ (a): We claim that $p \cdot 1$ is a unit in $A$ for every prime $p$ in $N$. To show this, choose an element $\mathbf{b}$ in $\mathrm{gh}_N(A)$ whose $p$th component equals $q^{p-1}a_1^p + 1$. By the surjectivity of $\Phi_N^q$ there exists a unique element $(x_n)_{n \in N}$ in $\mathbb{W}_N^q(A)$ such that $\Phi_N^q((x_n)_{n \in N}) = \mathbf{b}$. Comparing the $p$th component yields that

$$q^{p-1}x_1^p + px_p = q^{p-1}a_1^p + 1.$$

Since $x_1 = a_1$, this equality implies that $px_p = 1$. Consequently $p \cdot 1$ is a unit in $A$. Equivalently, $n \cdot 1$ is a unit in $A$ for every $n \in N$. It follows that we have an injective ring homomorphism

$$i : \mathbb{Z}_N \to A, \qquad \frac{n}{m} \mapsto nm^{-1}.$$

Thus $A$ is a $\mathbb{Z}_N$-algebra. $\square$

Let $A$ be a commutative ring with identity. In case where $\mathbb{W}_N^q(A)$ is not strictly isomorphic to $\mathbb{W}_N(A)$, one of the significant differences between them is that the former may not have the identity. In the following, we will give a necessary and sufficient condition so that $\mathbb{W}^q(A)$ should have the identity.

First, denote the identity, if exists, by

$$1_{W_N^q(A)} = (e_n)_{n\in N}.$$

Note that $e_n$'s can be determined inductively by solving the system of equations

$$\sum_{d|n} dq^{\frac{n}{d}-1} e_d^{\frac{n}{d}} = 1 \tag{4.6}$$

for all $n \in N$. Note that $e_1 = 1$. Also, if $p$ is a prime in $N$, then Eq. (4.6) reduces to

$$q^{p-1} + pe_p = 1. \tag{4.7}$$

In general, Eq. (4.7) is not the case unless $p$ and $q$ are mutually disjoint. However, in case where the prime divisors of $q$ in $N$ are units in $A$, one can find $e_p$ by letting

$$e_p = \begin{cases} p^{-1}(1-q^{p-1}) & \text{if } p \text{ is a unit,} \\ (1-q^{p-1})/p & \text{otherwise.} \end{cases}$$

Note that the second term $(1-q^{p-1})/p$ is well defined by virtue of the *Little theorem of Fermat*.

**Theorem 19.** *Let $A$ be a commutative ring with identity and $q$ an integer. Then the following statements are equivalent.*

(a) *For every prime divisor $p$ in $N$ dividing $q$, $p \cdot 1$ is a unit in $A$.*
(b) *$\mathbb{W}_N^q(A)$ has the identity.*

**Proof.** (a) $\Rightarrow$ (b): If follows from Corollary 17(b).

(b) $\Rightarrow$ (a): By the assumption there exists a unique $x_p \in A$ such that

$$q^{p-1} + px_p = 1 \tag{4.8}$$

for every prime $p \in N$ which divides $q$. Then, Eq. (4.8) can be written as

$$p \cdot \left(\frac{q^{p-1}}{p} + x_p\right) = 1.$$

This implies that $p \cdot 1$ is a unit. □

*4.2. Deformed nested necklace rings*

As before, let $A$ be a commutative ring with identity. In this section, we provide a necessary and sufficient condition of $A$ so that $Nr_N^q(A)$ and $Nr_N^r(A)$ should be strictly isomorphic. Indeed, all the results obtained in the previous section remain still true.

**Definition 20.** Let $q$ and $r$ be arbitrary integers. $Nr_N^q(A)$ is said to be *strictly isomorphic* to $Nr_N^r(A)$ if there exists a ring isomorphism, say $\mathfrak{n}_q^r : Nr_N^q(A) \to Nr_N^r(A)$, satisfying $\varphi_N^q = \varphi_N^r \circ \mathfrak{n}_q^r$. In this case, $\mathfrak{n}_q^r$ is called a *strict isomorphism*.

Following in the exactly same way as in Theorem 14 and Corollary 16, we can establish the following facts.

**Theorem 21.** *Fix two arbitrary integers $q$ and $r$. Let*

$$D^{\mathrm{pr}}(q) \cap N = \{p_1, \dots, p_k; c_1, \dots, c_s\},$$
$$D^{\mathrm{pr}}(r) \cap N = \{p_1, \dots, p_k; d_1, \dots, d_t\}.$$

*Then, there exists a unique strict isomorphism between $Nr_N^q(A)$ and $Nr_N^r(A)$ if and only if $A$ is a $\mathbb{Z}[\frac{1}{c_i}, \frac{1}{d_j}\colon 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t]$-algebra.*

**Corollary 22.** *Let $q$ and $r$ be arbitrary integers. Then, we have*:

(a) *$Nr^q(A)$ and $Nr^r(A)$ are isomorphic as filtered algebras associated with the operators $V_m$, $m \geqslant 1$ if they are strictly isomorphic.*
(b) *Assume that $A$ is an integral domain of characteristic zero. Then, $Nr^q(A)$ is strictly isomorphic to $Nr^r(A)$ if they are isomorphic as filtered rings associated with the operators $V_m$, $m \geqslant 1$.*

Also, one can verify in a routine way that other statements such as Corollary 17, Theorem 18 remain true if we replace $\mathbb{W}_N^q$ (respectively $\mathbb{W}_N^r$) by $Nr_N^q$ (respectively $Nr_N^r$). So, we omit the proof. We close this section by providing an analogue of Theorem 19.

**Theorem 23.** *Let $A$ be a commutative ring with identity and $q$ an integer. Then the following statements are equivalent.*

(a) *For every prime divisor $p$ in $N$ dividing $q$, $p \cdot 1$ is a unit in $A$.*
(b) *$Nr_N^q(A)$ has the identity.*

**Proof.** (a) $\Rightarrow$ (b): From Theorem 21 it follows that the prime divisors in $N$ dividing $q$ are all units in $A$ if and only if $Nr_N^q(A)$ is strictly isomorphic to $Nr_N(A)$. Since $Nr_N(A)$ has the identity we are done.

(b) $\Rightarrow$ (a): By assumption the following linear system of equations (over $\mathbb{Q}$)

$$\sum_{d|n} d q^{\frac{n}{d}-1} X_d = 1, \quad n \in N,$$

has a unique solution set. For our purpose we have to show that $X_n$, $n \in \mathbb{N}$ should be in $\mathbb{Z}_{q \cap N}$. Note that $\mathbb{W}_N^q(\mathbb{Z}_{q \cap N})$ has the identity by Theorem 19. Using the fact that $Nr^q(B)$ is isomorphic to $\mathbb{W}^q(B)$ if $B$ is a binomial ring (refer to [10,12]), we can conclude the desired result because $\mathbb{Z}_{q \cap N}$ is a binomial ring. This implies that $X_n$, $n \geqslant 1$, belong to $\mathbb{Z}_{q \cap N}$. Since the solution set is unique $A$ should be a $\mathbb{Z}_{q \cap N}$-algebra. $\square$

*4.3. Functorial property*

In this section, we establish isomorphisms of functors,

$$\mathbb{W}_N^q \circ \mathbb{W}_M^q \cong \mathbb{W}_{MN}^q \quad \text{and} \quad Nr_N^q \circ Nr_M^q \cong Nr_{MN}^q,$$

for coprime truncation sets $M$ and $N$. Note that if $M$ and $N$ are truncation sets, then $MN := \{mn\colon m \in M,\ n \in N\}$ is also a truncation set.

Let $A$ be any commutative ring. Applying the functoriality of $\Phi_N^q$ to the ring homomorphism $\Phi_N^q(A) : \mathbb{W}_N^q(A) \to \mathrm{gh}_N(A)$, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{W}_N^q(\mathbb{W}_M^q(A)) & \xrightarrow{\Phi_N^q(\mathbb{W}_M^q(A))} & \mathrm{gh}_N(\mathbb{W}_M^q(A)) \\
{\scriptstyle \mathbb{W}_N^q(\Phi_M^q(A))}\downarrow & & \downarrow{\scriptstyle \Phi_N^q(A)^M} \\
\mathbb{W}_N^q(\mathrm{gh}_M(A)) & \xrightarrow{\Phi_N^q(\mathrm{gh}_M(A))} & \mathrm{gh}_N(\mathrm{gh}_M(A)).
\end{array}$$

The resulting ring homomorphism

$$\Phi_{M,N}^q(A) : \mathbb{W}_N^q\big(\mathbb{W}_M^q(A)\big) \to \mathrm{gh}_N\big(\mathrm{gh}_M(A)\big)$$

sends $\mathbf{X} = (X_{m,n})_{m\in M,\, n\in N}$ to $(X_{(m,n)}^q)_{m\in M,\, n\in N}$, where

$$X_{(m,n)}^q = \sum_{d|n} dq^{\frac{n}{d}-1}\Bigg(\sum_{c|m} cq^{\frac{m}{c}-1} X_{c,d}^{\frac{m}{c}}\Bigg)^{\frac{n}{d}}.$$

In particular, if $M \cap N = \{1\}$, then $MN \cong M \times N$. It follows that $\mathrm{gh}_N(\mathrm{gh}_M(A)) = \mathrm{gh}_{MN}(A)$.

**Theorem 24.** *Let $q$ be any integer, and $M$, $N$ be truncation sets with $M \cap N = \{1\}$. Then there is a unique functorial isomorphism*

$$\omega_{M,N}^q : \mathbb{W}_N^q \circ \mathbb{W}_M^q \to \mathbb{W}_{MN}^q$$

*satisfying $\Phi_{M,N}^q = \Phi_{MN}^q \circ \omega_{M,N}^q$.*

**Proof.** The proof can be accomplished by following the method in [14, Proof of Theorem 1]. First, we recall that all three functors $\mathbb{W}_N^q \circ \mathbb{W}_M^q$, $\mathbb{W}_{MN}^q$, and $\mathrm{gh}_{MN}$ are represented by the polynomial ring

$$R := \mathbb{Z}[X_{mn}\colon m \in M,\ n \in N],$$

with variables $X_{mn}$. Set $\mathbf{X} = (X_{mn})_{m\in M,\, n\in N}$. By Yoneda's Lemma, a functorial map

$$\omega_{M,N}^q : \mathbb{W}_N^q \circ \mathbb{W}_M^q \to \mathbb{W}_{MN}^q$$

satisfying $\Phi^q_{M,N} = \Phi^q_{MN} \circ \omega^q_{M,N}$ corresponds to an element

$$\mathbf{Z} = (Z_{mn})_{m\in M,\, n\in N} \in \mathbb{W}^q_{MN}(R)$$

satisfying $\Phi^q_{M,N}(\mathbf{X}) = \Phi^q_{MN}(\mathbf{Z})$. Furthermore, $\omega^q_{M,N}$ will be bijective if and only if

$$R = \mathbb{Z}[Z_{mn}\colon\ m \in M,\ n \in N].$$

From the fact that $\Phi^q_{M,N}(R \otimes \mathbb{Q})$ and $\Phi^q_{MN}(R \otimes \mathbb{Q})$ are isomorphisms it follows

$$\mathbf{Z} = \big(\Phi^q_{MN}\big)^{-1}\big(\Phi^q_{M,N}(\mathbf{X})\big) \in \mathbb{W}^q_{MN}(R \otimes \mathbb{Q}).$$

Therefore $\omega^q_{M,N}$ will be a ring isomorphism and unique if exists. Observe that $Z_{mn} - X_{mn}$ belongs to

$$\mathbb{Q}[X_{cd}\colon\ c \mid m,\ d \mid n,\ \text{and}\ cd < mn].$$

Consequently, we are done if we can prove that $\mathbf{Z} \in \mathbb{W}^q_{MN}(R)$. To this end, let us consider the system of identities over $\mathbb{Q}$

$$\sum_{d|n} dq^{\frac{n}{d}-1}\left(\sum_{c|m} cq^{\frac{m}{c}-1} X_{cd}^{\frac{m}{c}}\right)^{\frac{n}{d}} = \sum_{d|mn} dq^{\frac{mn}{d}-1} Z_d^{\frac{mn}{d}}, \quad m \in M,\ n \in N. \tag{4.9}$$

Multiplying $q$ to both sides Eq. (4.9) yields

$$\sum_{d|n} d\left(\sum_{c|m} c(qX_{cd})^{\frac{m}{c}}\right)^{\frac{n}{d}} = \sum_{d|mn} d(qZ_d)^{\frac{mn}{d}}, \quad m \in M,\ n \in N.$$

Equivalently,

$$\Phi_{M,N}\big((qX_{mn})_{m\in M,\, n\in N}\big) = \Phi_{MN}\big((qZ_{mn})_{m\in M,\, n\in N}\big).$$

On the other hand, [14, Theorem 1] asserts that

$$qZ_{mn} - qX_{mn} \in \mathbb{Z}[qX_{cd}\colon\ c \mid m,\ d \mid n,\ cd < mn].$$

Since $Z_{mn}$ is a polynomial without constant term, we can conclude that

$$Z_{mn} - X_{mn} \in \mathbb{Z}[X_{cd}\colon\ c \mid m,\ d \mid n,\ cd < mn]$$

if $q \neq 0$. In case $q = 0$, then Eq. (4.9) reduces to

$$X_{mn} = Z_{mn}, \quad m \in M,\ n \in N.$$

So, we are done. $\square$

Next, we establish an isomorphism of functors $Nr_N^q \circ Nr_M^q \cong Nr_{MN}^q$. Applying the functoriality of $\varphi_N^q$ to the ring homomorphism $\varphi_M^q(A): Nr_M^q(A) \to \mathrm{gh}_M(A)$, we have the commutative diagram

$$\begin{tikzcd}
Nr_N^q(Nr_M^q(A)) \arrow[r, "\varphi_N^q(Nr_M^q(A))"] \arrow[d, "Nr_N^q(\varphi_M^q(A))"'] & \mathrm{gh}_N(Nr_M^q(A)) \arrow[d, "\varphi_N^q(A)^M"] \\
Nr_N^q(\mathrm{gh}_M(A)) \arrow[r, "\varphi_N^q(\mathrm{gh}_M(A))"] & \mathrm{gh}_N(\mathrm{gh}_M(A)).
\end{tikzcd}$$

The resulting ring homomorphism

$$\varphi_{M,N}^q(A): Nr_N^q\big(Nr_M^q(A)\big) \to \mathrm{gh}_N\big(\mathrm{gh}_M(A)\big)$$

sends $\mathbf{X} = (X_{mn})_{m\in M,\, n\in N}$ to $(X_{(m,n)}^q)_{m\in M,\, n\in N}$, where

$$X_{(m,n)}^q := \sum_{d|n} dq^{\frac{n}{d}-1}\Bigg(\sum_{c|m} cq^{\frac{m}{c}-1} X_{c,d}\Bigg).$$

**Theorem 25.** *Let $q$ be any integer, and $M, N$ be truncation sets with $M \cap N = \{1\}$. Then there is a unique functorial isomorphism*

$$\mathfrak{n}_{M,N}^q : Nr_N^q \circ Nr_M^q \to Nr_{MN}^q$$

*satisfying $\varphi_{M,N}^q = \varphi_{MN}^q \circ \mathfrak{n}_{M,N}^q$.*

**Proof.** Set $\mathbf{X} = (X_{mn})_{m\in M,\, n\in N}$. As in the proof of Theorem 24 it suffices to show that

$$\mathbf{Z} = (Z_{mn})_{m\in M,\, n\in N} := \big(\varphi_{MN}^q\big)^{-1}\big(\varphi_{M,N}^q(\mathbf{X})\big) \in Nr_{MN}^q(R), \tag{4.10}$$

where $R = \mathbb{Z}[X_{mn}\colon m \in M,\ n \in N]$. Note that Eq. (4.10) is identical to the system of identities

$$\sum_{d|n} dq^{\frac{n}{d}-1}\Bigg(\sum_{c|m} cq^{\frac{m}{c}-1} X_{cd}\Bigg) = \sum_{d|mn} dq^{\frac{mn}{d}-1} Z_d, \quad m \in M,\ n \in N.$$

Equivalently,

$$\sum_{\substack{c|m\\ d|n}} cdq^{\frac{m}{c}+\frac{n}{d}-2} X_{cd} = \sum_{\substack{c|m\\ d|n}} cdq^{\frac{mn}{cd}-1} Z_{cd}, \quad m \in M,\ n \in N.$$

On the other hand, since $R \otimes \mathbb{Q}$ is a binomial ring, we obtain the isomorphism

$$\tau_M^q(R\otimes\mathbb{Q}): \mathbb{W}_M^q(R\otimes\mathbb{Q}) \to Nr_M^q(R\otimes\mathbb{Q})$$

in view of Theorem 5. Induce the isomorphism

$$\mathbb{W}_N^q\big(\tau_M^q(R\otimes\mathbb{Q})\big):\mathbb{W}_N^q\big(\mathbb{W}_M^q(R\otimes\mathbb{Q})\big)\to\mathbb{W}_N^q\big(Nr_M^q(R\otimes\mathbb{Q})\big)$$

from $\tau_M^q(R\otimes\mathbb{Q})$, and then compose it with the isomorphism

$$\tau_N^q\big(Nr_M^q(R\otimes\mathbb{Q})\big):\mathbb{W}_N^q\big(Nr_M^q(R\otimes\mathbb{Q})\big)\to Nr_N^q\big(Nr_M^q(R\otimes\mathbb{Q})\big).$$

Finally, we can obtain the isomorphism

$$\tau_{M,N}^q\big(:=\tau_N^q\big(Nr_M^q(R\otimes\mathbb{Q})\big)\circ\mathbb{W}_N^q\big(\tau_M^q(R\otimes\mathbb{Q})\big)\big):\mathbb{W}_{M,N}^q(R\otimes\mathbb{Q})\to Nr_{M,N}^q(R\otimes\mathbb{Q}),$$

where the notation $\mathbb{W}_{M,N}^q(R\otimes\mathbb{Q})$ (respectively $Nr_{M,N}^q(R\otimes\mathbb{Q})$) represents the ring

$$\mathbb{W}_N^q\big(\mathbb{W}_M^q(R\otimes\mathbb{Q})\big)\quad\big(\text{respectively } Nr_N^q\big(Nr_M^q(R\otimes\mathbb{Q})\big)\big).$$

Observe the following commutative diagram

$$\begin{tikzcd}
\mathbb{W}_{M,N}^q(R\otimes\mathbb{Q}) \arrow[rr, "\tau_{M,N}^q"] \arrow[dr, "\Phi_{M,N}^q"] \arrow[dd, "\omega_{M,N}^q"'] & & Nr_{M,N}^q(R\otimes\mathbb{Q}) \arrow[dl, "\varphi_{M,N}^q"'] \arrow[dd, "\mathfrak{n}_{M,N}^q"] \\
& \mathrm{gh}_{MN}(R\otimes\mathbb{Q}) & \\
\mathbb{W}_{MN}^q(R\otimes\mathbb{Q}) \arrow[rr, "\tau_{MN}^q"] \arrow[ur, "\Phi_{MN}^q"'] & & Nr_{MN}^q(R\otimes\mathbb{Q}). \arrow[ul, "\varphi_{MN}^q"]
\end{tikzcd}$$

Here, $\mathfrak{n}_{M,N}^q$ is defined by the composition map $(\varphi_{MN}^q)^{-1}\circ\varphi_{M,N}^q$, which equals $\tau_{M,N}^q\circ\omega_{M,N}^q\circ(\tau_{MN}^q)^{-1}$ by the commutativity. As in the proof of Theorem 24, $\mathfrak{n}_{M,N}^q$ is determined by the polynomials $\{Z_{mn}\colon m\in M,\ n\in N\}$ under the mapping

$$\mathfrak{n}_{M,N}^q:(X_{mn})_{m\in M,\,n\in N}\mapsto(Z_{mn})_{m\in M,\,n\in N}.$$

We claim that

$$Z_{mn}-X_{mn}\in\mathbb{Z}[X_{cd}\colon c\mid m,\ d\mid n,\ cd<mn] \tag{4.11}$$

for all $m\in M$, $n\in N$. Suppose not. Then, for some $m\in M$, $n\in N$, $Z_{mn}$ contains a term

$$c_{cd}X_{cd},\quad \text{where } c\mid m,\ d\mid n,\ cd<mn,$$

whose coefficient $c_{cd}$ is not an integer. Specializing $X$ by the vector $E_{c,d}:=(e_{a,b})_{a\in M,\,b\in N}$ given by

$$e_{ab}:=\begin{cases}1 & \text{if } a=c,\ b=d,\\ 0 & \text{otherwise}\end{cases}$$

implies that the $mn$th component of $\mathfrak{n}^q_{M,N}(E_{c,d})$, which equals $c_{cd}$, does not belong to $\mathbb{Z}$. But, this gives rise to a contradiction since

$$\mathfrak{n}^q_{M,N}\big(Nr_{M,N}(\mathbb{Z})\big) = Nr^q_{MN}(\mathbb{Z}).$$

This is because $\mathbb{Z}$ is a binomial ring. In conclusion, Eq. (4.11) is the case for all $m \in M$, $n \in N$. This completes the proof. □

**Example 26.** Let us compute polynomials $Z_{pp'}$ for primes $p \in M$ and $p' \in N$. Clearly, $Z_{m1} = X_{m1}$ and $Z_{1n} = X_{1n}$ for all $m \in M$, $n \in N$. Then we have

$$Z_{pp'} = X_{pp'} + \frac{p^{p'-1} - 1}{p'} X_{p1}^{p'} + \frac{1}{pp'} \sum_{r=1}^{p'-1} p^r \binom{p'}{r} X_{11}^{p(p'-r)} X_{p1}^r.$$

Note that all the coefficients are integers.

## 5. Applications

### 5.1. Generalized Möbius function

In this section, we study how the generalized Möbius function, $\mu_N$, due to E. Cohen [2], arises naturally in our frame. Throughout this section $N$ will denote a monoidal truncation set. Let us first consider a ring homomorphism

$$\begin{aligned} \varphi^{1,0}_{N,N^\perp} : Nr^0_{N^\perp}\big(Nr_N(A)\big) &\to \mathrm{gh}(A), \\ (a_n)_{n\in\mathbb{N}} &\mapsto \Bigg(\sum_{\substack{d|n \\ d\in N}} \frac{n}{d} a_{\frac{n}{d}}\Bigg)_{n\in\mathbb{N}}, \end{aligned}$$

which can be induced by the functorial rule. Note that $\varphi^{1,0}_{N,N^\perp}$ factors through as follows:

$$Nr^0_{N^\perp}\big(Nr_N(A)\big) \xrightarrow{Nr^0_{N^\perp}(\varphi_N(A))} Nr^0_{N^\perp}\big(\mathrm{gh}_N(A)\big) \xrightarrow{\varphi^0_{N^\perp}(\mathrm{gh}_N(A))} \mathrm{gh}_{N^\perp}\big(\mathrm{gh}_N(A)\big) \cong \mathrm{gh}(A).$$

It is easy to show that the following diagram

$$\begin{tikzcd} Nr_N(A) \arrow[r, "\iota"] \arrow[d, "\varphi_N"] & Nr^0_{N^\perp}(Nr_N(A)) \arrow[d, "\varphi^{1,0}_{N,N^\perp}"] & Nr^0_{N^\perp}(A) \arrow[l, "\iota"'] \arrow[d, "\varphi^0_{N^\perp}"] \\ \mathrm{gh}_N(A) \arrow[r, "\iota"] & \mathrm{gh}(A) & \mathrm{gh}_{N^\perp}(A) \arrow[l, "\iota"'] \end{tikzcd}$$

is commutative. Let $A$ be a $\mathbb{Q}$-algebra to make sure that the inverse of $\varphi^{1,0}_{N,N^\perp}$ exists. As a preparatory step to describe the inverse of $\varphi^{1,0}_{N,N^\perp}$ we would like to recall the generalized Möbius function $\mu_N$. By definition

$$\mu_N(n) = \sum_{\substack{d|n \\ d\in N^\perp}} \mu\left(\frac{n}{d}\right).$$

It is well known [1] that $\mu_N$ is a unique arithmetical function satisfying

$$\sum_{\substack{d|n \\ d\in N}} \mu_N\left(\frac{n}{d}\right) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 27.** [1,2]

(a) *Let $g$ be a multiplicative function. If $f$ is such that*

$$g(n) = \sum_{\substack{d|n \\ d\in N}} f\left(\frac{n}{d}\right),$$

*then $f$ is multiplicative. In this case, we have*

$$f(n) = \sum_{d|n} g(d)\mu_N\left(\frac{n}{d}\right).$$

(b) *Let $f$ and $g$ be multiplicative functions such that $g(1) \neq 0$. If*

$$g(n) = \sum_{\substack{d|n \\ d\in N}} f\left(\frac{n}{d}\right) \quad \Longleftrightarrow \quad f(n) = \sum_{d|n} g(d)\mu^*\left(\frac{n}{d}\right),$$

*then $\mu^* = \mu_N$.*

Given $m \in \mathbb{N}$, let us define a corresponding matrix, $\zeta_{m,N}$, on the lattice $D(m)$ of divisors of $m$, by

$$\zeta_{m,N}(d_1, d_2) = \begin{cases} 1 & \text{if } d_1 = \frac{d_2}{e}, \text{ where } e \in N \cap D(m), \\ 0 & \text{otherwise,} \end{cases}$$

and denote by $\mu_{m,N}$ its inverse. It is obvious that the $m$th component of $\varphi^{1,0}_{N,N^\perp}((a_n)_{n\in\mathbb{N}})$ is given by the $m$th component of $(a_d)_{d\in D(m)}\zeta_{m,N}$. Consequently, one can verify that

$$\begin{aligned} \left(\varphi^{1,0}_{N,N^\perp}\right)^{-1} : \mathrm{gh}(A) &\to Nr^0_{N^\perp}\left(Nr_N(A)\right), \\ (b_n)_{n\in\mathbb{N}} &\mapsto \left(\frac{1}{n}\sum_{d|n} \mu_{n,N}(d,n) b_d\right)_{n\in\mathbb{N}}. \end{aligned}$$

**Theorem 28.** *If $d$ is a divisor of $n$, then $\mu_{n,N}(d,n) = \mu_N(\frac{n}{d})$.*

**Proof.** Let $f$ be a multiplicative function such that $f(1) \neq 0$. Set $a_n = \frac{f(n)}{n}$ for every $n \in \mathbb{N}$. From

$$b_n = \sum_{\substack{d|n \\ d \in N}} \frac{n}{d} a_{\frac{n}{d}} \quad \Longleftrightarrow \quad a_n = \frac{1}{n} \sum_{d|n} \mu_{n,N}(d,n) b_d,$$

it follows that

$$b_n = \sum_{\substack{d|n \\ d \in N}} f\left(\frac{n}{d}\right) \quad \Longleftrightarrow \quad f(n) = \sum_{d|n} \mu_{n,N}(d,n) b_d.$$

Hence, Lemma 27(b) implies the desired result. □

In particular, if

$$b_n = \begin{cases} x^n & \text{if } n \in N, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\frac{1}{n} \sum_{d|n} \mu_N\left(\frac{n}{d}\right) x^d = \begin{cases} M(x,n) & \text{if } n \in N, \\ 0 & \text{otherwise.} \end{cases}$$

### 5.2. *Polynomials similar to $q$-necklace polynomials*

In this section, we will introduce polynomials similar to $q$-necklace polynomials $M^q(x,n)$ (see Eq. (2.2)), which arise from our framework very naturally. Recall that

$$M^q(x) = \left(M^q(x,n)\right)_{n \in \mathbb{N}} = \tau^q\big((x,0,0,\ldots)\big).$$

Now, for each positive integer $n$ we let

$$N^q(x,n) := \left(\tau^q\right)^{-1}\big((x,0,0,\ldots)\big).$$

It can be easily verified that it is a polynomial in $q, x$ with rational coefficients whose generating function is given by

$$q^{n-1} x = \sum_{d|n} d q^{\frac{n}{d}-1} N^q(x,d)^{\frac{n}{d}}.$$

Multiply $q$ to both sides to obtain the relation

$$\left(\frac{1}{1-qt}\right)^x = \prod_{n=1}^{\infty} \frac{1}{1 - qN^q(x,n)t^n}. \tag{5.1}$$

**Theorem 29.** *For every $q \in \mathbb{Z}$ and $n \in \mathbb{N}$, $N^q(x,n)$ is a polynomial in $x$ of degree $n$ which takes on integer values for integer arguments.*

**Proof.** Since $\mathbb{Z}$ is a binomial ring, $\tau^q : \mathbb{W}^q(\mathbb{Z}) \to Nr^q(\mathbb{Z})$ is a ring isomorphism. This implies that if $x \in \mathbb{Z}$, then every component of the vector $(\tau^q)^{-1}(x, 0, 0, \ldots)$ should be an integer. So, we are done. □

If we replace $t$ by $-t$, then Eq. (5.1) can be written as

$$(1+qt)^x = \prod_{n=1}^{\infty} \bigl(1 + q\tilde{N}^q(x,n)t^n\bigr),$$

where $\tilde{N}^q(x,n) := (-1)^{n+1} N^q(x,n)$. One can see immediately that $\tilde{N}^q(x,n)$ also takes on integer values for integer arguments.

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